Microeconomics C:
Exam January 2012 - Solutions

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1. (a) Solve the normal-form game below by iterated elimination of strictly dominated strategies. Describe briefly each step.

	D	E	F	G	H
A	$8,5$	$1,6$	$3,6$	$5,4$	$6,7$
B	$7,4$	$9,2$	$2,3$	$6,3$	$4,3$
C	$6,8$	$2,1$	$4,3$	$6,7$	$5,4$

SOLUTION: First, E is (strictly) dominated by H and G is dominated by D for player 2. Then B is dominated by A for player 1. Then F is dominated by H for player 2. Then C is dominated by A for player 1. Finally, D is dominated by H for player 2. Thus the only strategy profile that survives iterated elimination of strictly dominated strategies is (A, H).

(b) Consider the following normal-form game:

	L	R
U	$4,2$	$2,3$
D	$1,4$	$3,1$

Find the mixed strategy Nash equilibrium. In this equilibrium, what is the probability that the outcome will be (U, R)? Calculate the expected utility for player 1 in the equilibrium.

SOLUTION: Let p denote the probability that player 1 plays U and let q denote the probability that player 2 plays L. In a mixed NE, each player is indifferent between the pure strategies he uses with positive probability. Thus we get the following equations:

$$\begin{aligned} 4q + 2(1-q) &= q + 3(1-q) \\ 2p + 4(1-p) &= 3p + (1-p) \end{aligned}$$

From these equations we get the mixed NE:

$$p^* = \frac{3}{4} \text{ and } q^* = \frac{1}{4}.$$

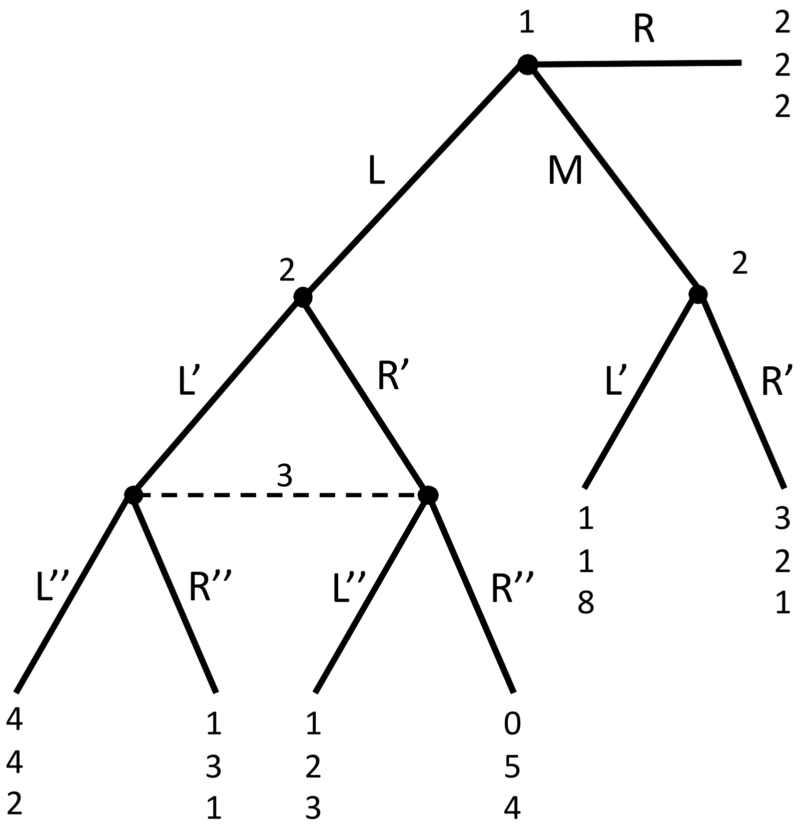
I.e., player 1 plays U with probability $\frac{3}{4}$ and player 2 plays L with probability $\frac{1}{4}$.
The probability that the outcome is (U, R) is

$$p^* \cdot (1 - q^*) = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}.$$

The expected utility for player 1 in the equilibrium is

$$\begin{aligned} p^*q^* \cdot 4 + p^*(1-q^*) \cdot 2 + (1-p^*)q^* \cdot 1 + (1-p^*)(1-q^*) \cdot 3 \\ = \frac{3}{16} \cdot 4 + \frac{9}{16} \cdot 2 + \frac{1}{16} \cdot 1 + \frac{3}{16} \cdot 3 = \frac{40}{16} = \frac{5}{2}. \end{aligned}$$

2. Consider the dynamic game given by the game tree below:



(a) Is it a game of perfect or imperfect information? How many subgames are there in the game (do not count the game itself as a subgame)? Write down the set of strategies for each player.

SOLUTION: Player 3 's information set consists of two decision nodes, so it is a game of imperfect information. There are two subgames, one starting at each of player 2's decision nodes. The strategy sets for the players are:

$$
\begin{aligned}
S_1 &= \{L, M, R\} \\
S_2 &= \{L'L', L'R', R'L', R'R'\} \\
&\quad \text{(for example, } L'R' \text{ means } L' \text{ at left node, } R' \text{at right node)} \\
S_3 &= \{L'', R''\}
\end{aligned}
$$

(b) Find all pure strategy subgame perfect Nash equilibria.

SOLUTION: First, find all pure strategy NE in the two subgames. In the left subgame there are two NE: (L', L'') and (R', R''). In the right subgame player 2 plays R'. If player 2 and 3 play (L', L'') in the left subgame, player 1 will play L. If 2 and 3 play (R', R''), 1 will play M. Thus we have found two pure strategy SPNE:

$$(L, L'R', L'') \text{ and } (M, R'R', R'').$$

(c) Find a pure strategy Nash equilibrium that is not subgame perfect and in which player 1 plays L.

SOLUTION: If player 1 plays L then the only subgame that is off the equilibrium path is the one to the right. Since we want a NE that is not subgame perfect, player 2 has to play L' in this subgame. The left subgame is on the equilibrium path, so player 2 and 3 has to play a NE. And since it has to be optimal for player 1 to play L, the only possibility is that they play (L', L''). Thus we have the following NE that is not subgame perfect:

$$(L, L'L', L'').$$

3. Two profit maximizing firms ($i = 1, 2$) produce a homogeneous product. The quantity produced by firm i is $q_i \geq 0$. The inverse demand function is

$$P(Q) = 12 - x \cdot Q,$$

where $Q = q_1 + q_2$ and $x > 0$ is a constant. The marginal cost of firm 1 is $c \in (0, 6)$. Firm 2's marginal cost is 1. There are no fixed costs.

(a) Write down the profit functions and derive the best response functions for the two firms.

SOLUTION: The profit functions for the two firms are

$$\begin{aligned}
\pi_1(q_1, q_2) &= (12 - x(q_1 + q_2))q_1 - cq_1 \\
\pi_2(q_1, q_2) &= (12 - x(q_1 + q_2))q_2 - q_2
\end{aligned}$$

Derive the first order conditions to get the following best response functions:

$$\begin{aligned}
BR_1(q_2) &= \frac{12 - c - xq_2}{2x} \\
BR_2(q_1) &= \frac{11 - xq_1}{2x}
\end{aligned}$$

(b) Let $x = 1$. Find the Nash equilibrium of the game where the firms choose their quantities simultaneously and independently. How do the equilibrium quantities of the two firms depend on c? Give an intuitive explanation of your findings.

SOLUTION: In a NE each firm best responds to the quantity of the other firm. Thus the conditions for (q_1^*, q_2^*) to be a Nash equilibrium are:

$$\begin{aligned}
q_1^* &= \frac{12 - c - xq_2^*}{2x} \\
q_2^* &= \frac{11 - xq_1^*}{2x}
\end{aligned}$$

Let $x = 1$ and solve for q_1^* and q_2^* to get the NE:

$$\begin{aligned}
q_1^* &= \frac{13 - 2c}{3} \\
q_2^* &= \frac{10 + c}{3}
\end{aligned}$$

We see that q_1^* depends negatively on c while q_2^* depends positively on c. The reason for the first finding is obvious: with a higher marginal cost, firm 1 will produce less. And when firm 1 produces less, the market price increases, which makes firm 2 produce more.

(c) [*Note: This is the longest question of the exam. So you may want to do other questions first.*]

Let $x = 1$ and $c = 2$. Suppose the game between the two firms is repeated over an infinite time horizon $t = 1, 2, ..., \infty$. The discount factor of each firm is $\delta \in (0, 1)$. In this infinitely repeated game, specify trigger strategies such that the outcome of each stage is $(q_1, q_2) = (2, 3)$. Find the inequalities that must be satisfied for the trigger strategies to constitute a subgame perfect Nash equilibrium. Find the lowest value of δ such that the inequalities are satisfied.

SOLUTION: With $x = 1$ and $c = 2$ the stage game NE is (see question b):

$$\begin{aligned} q_1^{NE} &= 3 \\ q_2^{NE} &= 4 \end{aligned}$$

Consider the following trigger strategies:

- Player 1:
  - If $t = 1$ or if the outcome of all previous stages was $(q_1, q_2) = (2, 3)$, choose $q_1 = 2$
  - Otherwise, choose $q_1 = 3$
- Player 2:
  - If $t = 1$ or if the outcome of all previous stages was $(q_1, q_2) = (2, 3)$, choose $q_2 = 3$
  - Otherwise, choose $q_1 = 4$

If the players play these strategies, the outcome of all stages will be $(q_1, q_2) = (2, 3)$. To find the inequalities that must be satisfied for the strategies to constitute a SPNE, first note that the optimal one-shot deviations are (use the best response functions from question a):

$$\begin{aligned} q_1^D &= \frac{12 - 2 - 3}{2} = \frac{7}{2} \\ q_2^D &= \frac{11 - 2}{2} = \frac{9}{2} \end{aligned}$$

We can then calculate the per period profits of each for the two firms in the normal phase, in the stage game NE, and after the optimal one-shot deviation:

$$\begin{aligned}
\pi_1^{Normal} &= (12-(2+3))2-2\cdot 2 = 10\\
\pi_1^{NE} &= (12-(3+4))3-2\cdot 3 = 9\\
\pi_1^{D} &= (12-(\frac{7}{2}+3))\frac{7}{2}-2\cdot\frac{7}{2} = \frac{49}{4}
\end{aligned}$$

$$\begin{aligned}
\pi_2^{Normal} &= (12-(2+3))3-3 = 18\\
\pi_2^{NE} &= (12-(3+4))4-4 = 16\\
\pi_2^{D} &= (12-(2+\frac{9}{2}))\frac{9}{2}-\frac{9}{2} = \frac{81}{4}
\end{aligned}$$

For the trigger strategies to constitute a SPNE, the following inequality must be satisfied for $i=1,2$:

$$\sum_{t=1}^{\infty}\delta^{t-1}\pi_i^{Normal} \geq \pi_i^D + \sum_{t=2}^{\infty}\delta^{t-1}\pi_i^{NE}.$$

These inequalities represent the requirement that if $t=1$ or the outcome of all previous stages was $(q_1,q_2)=(2,3)$ then it should be optimal for each firm to play as specified by the trigger strategy when the other firm also uses its trigger strategy. (In all other subgames the trigger strategies trivially constitute a Nash equilibrium). Plugging the profits into the inequalities we get:

$$\begin{aligned}
\frac{10}{1-\delta} &\geq \frac{49}{4}+\frac{9\delta}{1-\delta} \text{ and}\\
\frac{18}{1-\delta} &\geq \frac{81}{4}+\frac{16\delta}{1-\delta}.
\end{aligned}$$

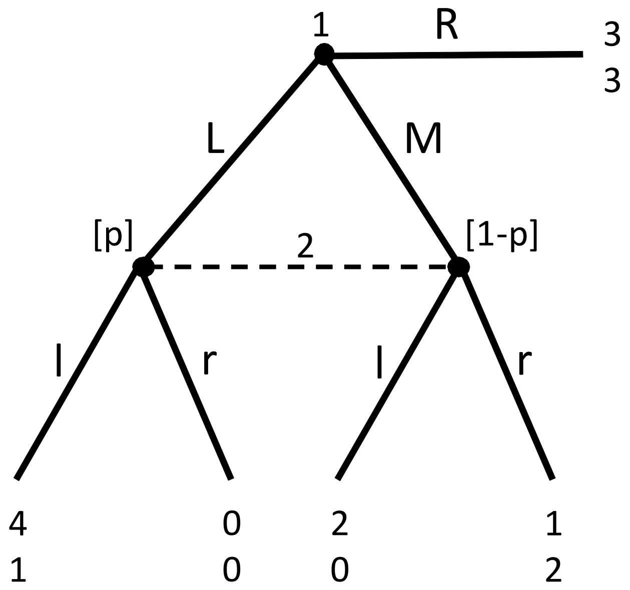
Multiply by $4(1-\delta)$ and isolate δ to get

$$\delta \geq \frac{9}{13} \text{ and } \delta \geq \frac{9}{17}$$

Thus the trigger strategies constitute a SPNE precisely if

$$\delta \geq \frac{9}{13}$$

4. Consider the game given by the game tree below:



(a) Find all pure strategy Nash equilibria and perfect Bayesian equilibria.

SOLUTION: (Note that the game is quite similar to the one in figure 4.4.1, p. 233 in Gibbons.)

An easy way to find all pure strategy NE is to write out the normal-form of the game as a bi-matrix:

	l	r
L	$\underline{4},\underline{1}$	0,0
M	2,0	$1,\underline{2}$
R	$3,\underline{3}$	$\underline{3},\underline{3}$

I.e., the pure strategy NE of the game are:

$$(L, l) \text{ and } (R, r)$$

To have a PBE, we have the extra requirements that player 2 has to choose optimally given some belief $p \in [0, 1]$ and that, on the equilibrium path, p has to be determined by Bayes rule and player 1's strategy (Requirement 1-3, see p.177-8 in Gibbons). For player 2, l is optimal when $p \geq \frac{2}{3}$ and r is optimal when $p \leq \frac{2}{3}$.

We therefore get the following pure strategy PBE:

$$(L, l, p = 1) \text{ and}$$
$$(R, r, p) \text{ for all } p \leq \frac{2}{3}$$

(b) For each of the perfect Bayesian equilibria you found in question (a), determine whether it satisfies Requirement 5 from section 4.4 in Gibbons. Explain.

SOLUTION: Requirement 5 says that players' beliefs should not put any probability mass on other players playing a strategy that is strictly dominated starting at some information set (see Gibbons p. 235). In this game, M is strictly dominated by R for player 1. Thus requirement 5 gives us that player 2 should, given that his information set is reached, believe with certainty that player 1 has chosen L. I.e., he should set $p = 1$. Thus all the PBE (R, r, p), $p \leq \frac{2}{3}$ do not satisfy Requirement 5. $(L, l, p = 1)$ does satisfy Requirement 5.